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Reducing the maximum degree of a graph: comparisons of bounds

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Abstract

Let $\lambda(G)$ be the smallest number of vertices that can be removed from a non-empty graph G so that the resulting graph has a smaller maximum degree. Let $\lambda_{\rm e}(G)$ be the smallest number of edges that can be removed from G for the same purpose. Let k be the maximum degree of G, let t be the number of vertices of degree k, let $M(G)$ be the set of vertices of degree k, let n be the number of vertices in the closed neighbourhood of $M(G)$, and let m be the number of edges that have at least one vertex in $M(G)$. Fenech and the author showed that $\lambda(G) \leq \frac{n+(k-1)t}{2k}$, and they essentially showed that $\lambda(G) \leq n\left(1 - \frac{k}{k+1}\left(\frac{n}{(k+1)t}\right)^{1/k}\right)$. They also showed that $\lambda_{\rm e}(G) \leq \frac{m+(k-1)t}{2k-1}$ and that if $k \geq 2$, then $\lambda_{\rm e}(G) \leq m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{1/(k-1)}\right)$. These bounds are attained if G is the union of pairwise vertex-disjoint $(k+1)$-vertex stars. In this paper, we determine the cases in which one bound on $\lambda(G)$ is better than the other, and we show that the first bound on $\lambda_{\rm e}(G)$ is better than the second. This work is motivated by the likelihood that similar pairs of bounds will be discovered for other graph parameters and the same analysis can be applied.

1 Introduction

For basic terminology and notation in graph theory, we refer the reader to [2, 6]. The definitions of terms and notations used here are given in the papers [3, 5], which are the basis of the work presented here.

The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by $[n]$. For a set X, the set of 2-element subsets of X is denoted by $\binom{X}{2}$. Arbitrary sets are taken to be finite.

Every graph G is taken to be *simple*, that is, its vertex set $V(G)$ and edge set $E(G)$ satisfy $E(G) \subseteq \binom{V(G)}{2}$. We may represent an edge $\{v, w\}$ by vw. For $v \in V(G)$, $N_G(v)$ denotes $\{w \in V(G) \colon vw \in E(G)\}$, $N_G[v]$ denotes $N_G(v) \cup \{v\}$, $E_G(v)$ denotes $\{e \in E(G) \colon v \in e\}$, and $d_G(v)$ denotes $|N_G(v)|$ $(= |E_G(v)|)$ and is called the *degree of* v. For $X \subseteq V(G)$, $\bigcup_{v \in X} N_G[v]$ is denoted by $N_G[X]$ and is called the *closed neighbourhood of* X. The *maximum degree of* G is $\max\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\Delta(G)$. The set of vertices of G of degree $\Delta(G)$ is denoted by $M(G)$. For $X \subseteq V(G)$, $G[X]$ denotes the *subgraph of G induced by X*, that is, $G[X] = (X, E(G) \cap \binom{X}{2})$. For $R \subseteq V(G)$, $G - R$ denotes the subgraph of G obtained by removing the vertices in R from G, that is, $G - R = G[V(G) \backslash R]$. For $L \subseteq E(G)$, $G - L$ denotes the subgraph of G obtained by removing the edges in L from G, that is, $G - L = (V(G), E(G) \backslash L)$.

We call a subset R of $V(G)$ a *Δ-reducing set of G* if $\Delta(G - R) < \Delta(G)$ or $R = V(G)$ (note that $V(G)$ is the smallest Δ-reducing set of G if and only if $E(G) = \emptyset$). We call a subset L of $E(G)$ a *Δ-reducing edge set of G* if $\Delta(G - L) < \Delta(G)$ or $L = E(G) = \emptyset$. We denote the size of a smallest Δ-reducing set of G by $\lambda(G)$, and we denote the size of a smallest Δ-reducing edge set of G by $\lambda_{\rm e}(G)$.

Let $G_{\rm v}$ denote the subgraph of G induced by $N_G[M(G)]$, and let $G_{\rm e}$ denote the subgraph of G with vertex set $N_G[M(G)]$ and edge set $\bigcup_{v \in M(G)} E_G(v)$. As explained in [3, 5], we clearly have

$$\Delta(G_{\mathrm{v}}) = \Delta(G), \quad M(G_{\mathrm{v}}) = M(G), \quad \lambda(G_{\mathrm{v}}) = \lambda(G), \tag{1}$$

$$\Delta(G_{\mathrm{e}}) = \Delta(G), \quad M(G_{\mathrm{e}}) = M(G), \quad \lambda_{\mathrm{e}}(G_{\mathrm{e}}) = \lambda_{\mathrm{e}}(G), \tag{2}$$

$$|V(G_{\mathrm{v}})| \leq \sum_{v \in M(G)} |N_G[v]| = (\Delta(G)+1)|M(G)|, \tag{3}$$

$$|E(G_{\mathrm{e}})| \leq \sum_{v \in M(G)} |E_G(v)| = \Delta(G)|M(G)|. \tag{4}$$

By the handshaking lemma and (2), $2|E(G_{\mathrm{e}})| = \sum_{v \in V(G_{\mathrm{e}})} d_{G_{\mathrm{e}}}(v) \geq \sum_{v \in M(G_{\mathrm{e}})} d_{G_{\mathrm{e}}}(v) = \Delta(G_{\mathrm{e}})|M(G_{\mathrm{e}})| = \Delta(G)|M(G)|$, so

$$|E(G_{\mathrm{e}})| \geq \Delta(G)|M(G)|/2. \tag{5}$$

The graph parameters $\lambda(G)$ and $\lambda_{\mathrm{e}}(G)$ were investigated in [3] and [5], respectively. For each of them, two main general bounds were obtained, and the bounds are reached if, for example, G is the union of pairwise vertex-disjoint copies of a star (see [3, 4, 5]).

The following is the first main general bound proved in [3].

Theorem 1.1 ([3]). *If G is a graph, $k = \Delta(G) \geq 1$, $t = |M(G)|$, and $n = |V(G_{\mathrm{v}})|$, then*

$$\lambda(G) \leq \frac{n + (k-1)t}{2k}.$$

In [3], the result is actually stated with $n = |V(G)|$, but the improvement given by $n = |V(G_{\mathrm{v}})|$ is immediately deduced from (1). The extremal structures are determined in [4]. For $0 \leq p \leq 1$, let $u(p) = np + t(1-p)^{k+1}$. Using a probabilistic argument similar to that used by Alon in [1], it was also shown in [3] ([3, Proof of Theorem 2.7]) that

$$\lambda(G) \leq u(p) \text{ for any real number } p \text{ such that } 0 \leq p \leq 1, \tag{6}$$

and that this yields the bound

$$\lambda(G) \leq \frac{n \ln(k+1) + t}{k+1}.$$

However, by differentiating u with respect to p, we find that the minimum value of u occurs at $p = 1 - (\frac{n}{(k+1)t})^{1/k}$ (note that this satisfies $0 \leq p \leq 1$ by (3)), and hence it is $n(1 - (\frac{n}{(k+1)t})^{1/k}) + t(\frac{n}{(k+1)t})^{1+1/k} = n(1 - \frac{k}{k+1}(\frac{n}{(k+1)t})^{1/k})$. Thus, by (6), the following was essentially established in [3].

Theorem 1.2. *If G is a graph, $k = \Delta(G) \geq 1$, $t = |M(G)|$, and $n = |V(G_{\mathrm{v}})|$, then*

$$\lambda(G) \leq n\left(1 - \frac{k}{k+1}\left(\frac{n}{(k+1)t}\right)^{1/k}\right).$$

The following are the two main general bounds proved in [5].

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Theorem 1.3 ([5]). *If G is a graph, $k = \Delta(G) \geq 1$, $t = |M(G)|$, and $m = |E(G_{\mathrm{e}})|$, then*

$$\lambda_{\mathrm{e}}(G) \leq \frac{m + (k-1)t}{2k-1}.$$

Theorem 1.4 ([5]). *If G is a graph, $k = \Delta(G) \geq 2$, $t = |M(G)|$, and $m = |E(G_{\mathrm{e}})|$, then*

$$\lambda_{\mathrm{e}}(G) \leq m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{1/(k-1)}\right).$$

Theorem 1.4 was obtained by a probabilistic argument similar to that for Theorem 1.2.

In this paper, we determine the cases in which one bound on $\lambda(G)$ is better than the other, and we show that the bound in Theorem 1.3 is better than the bound in Theorem 1.4. This work is motivated by the likelihood that similar pairs of bounds will be discovered for other graph parameters and the same analysis can be applied.

For k, t, and n as in Theorems 1.1 and 1.2, let $b_1(k,t,n)$ and $b_2(k,t,n)$ be the bound in Theorem 1.1 and the bound in Theorem 1.2, respectively; that is,

$$b_1(k,t,n) = \frac{n + (k-1)t}{2k} \quad \text{and} \quad b_2(k,t,n) = n\left(1 - \frac{k}{k+1}\left(\frac{n}{(k+1)t}\right)^{1/k}\right).$$

By (3), $n \leq (k+1)t$, and equality holds if G is the union of t pairwise vertex-disjoint $(k+1)$-vertex stars, in which case the two bounds are equal and attained. We now consider $n < (k+1)t$. If $k = 1$, then we trivially have $b_1(k,t,n) < b_2(k,t,n)$. For $k \geq 2$, we have the following result, proved in Section 2.

Theorem 1.5. *Let k, t, and n be as in Theorems 1.1 and 1.2. Suppose $k \geq 2$ and $n < (k+1)t$. There exists a unique real number x_0 such that $2.438 < x_0 < \frac{2k^2}{k-1} + 1$ and $\left(\frac{2k^2}{2k^2-(k-1)(x_0-1)}\right)^k = x_0$.*
(a) If $n = \frac{(k+1)t}{x_0}$, then $b_1(k,t,n) = b_2(k,t,n)$.
(b) If $n > \frac{(k+1)t}{x_0}$, then $b_1(k,t,n) < b_2(k,t,n)$.
(c) If $n < \frac{(k+1)t}{x_0}$, then $b_1(k,t,n) > b_2(k,t,n)$.

Since n can be at most $(k+1)t$, this result tells us that the range of values of n for which the bound in Theorem 1.1 is better than the bound in Theorem 1.2 is wider than that for which the opposite holds.

For k, t, and m as in Theorem 1.4, let $b_3(k,t,m)$ and $b_4(k,t,m)$ be the bound in Theorem 1.3 and the bound in Theorem 1.4, respectively; that is,

$$b_3(k,t,m) = \frac{m + (k-1)t}{2k-1} \quad \text{and} \quad b_4(k,t,m) = m\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{1/(k-1)}\right).$$

In Section 3, we prove the following result.

Theorem 1.6. *If k, t, and m are as in Theorem 1.4, then*

$$b_3(k,t,m) \leq b_4(k,t,m).$$

Moreover, the inequality is strict if $m < kt$.

We now start working towards the proofs of Theorems 1.5 and 1.6. We use several well-known results from real analysis. The set of real numbers is denoted by $\mathbb{R}$, and the set of positive real numbers is denoted by $\mathbb{R}^+$. We make use of standard notation for real intervals. The base of the natural logarithm is denoted by e, that is, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718...$. We use x_1 to denote the real number 3.512... such that $e^{(x_1-1)/2} = x_1$.

2 Proof of Theorem 1.5

This section is dedicated to the proof of Theorem 1.5 and to the proof of the following improvement of the inequality for x_0 in Theorem 1.5.

Proposition 2.1. *Let $\delta \in \mathbb{R}^+$, and let k and x_0 be as in Theorem 1.5. Then, $x_0 > x_1 - \delta$ if k is sufficiently large.*

Lemma 2.2. *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by*

$$f(x) = \left(1+\frac{1}{x}\right)^{x+1}$$

for $x > 0$, then $f(x)$ decreases as x increases, and $\lim_{x\to\infty} f(x) = e$.

Proof. Let $g : (-\frac{1}{2}, \infty) \to \mathbb{R}$ be the function given by

$$g(z) = z - \ln(1+z)$$

for $z > -\frac{1}{2}$. The derivative $\frac{\mathrm{d}g}{\mathrm{d}z}$ is $1 - \frac{1}{1+z}$, which is negative for $-\frac{1}{2} < z < 0$, 0 for $z = 0$, and positive for $z > 0$. Thus, $g(z)$ increases from $g(0) = 0$ as z increases from 0 to infinity, and hence

$$g(z) > 0 \quad \text{for } z > 0. \tag{7}$$

We have $\ln f(x) = (x+1)\ln\left(1+\frac{1}{x}\right)$. Using implicit differentiation, we obtain $\frac{1}{f(x)}\frac{\mathrm{d}f}{\mathrm{d}x} = \ln\left(1+\frac{1}{x}\right) + (x+1)\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}$. Thus, by (7) with $z = \frac{1}{x}$, $-\frac{1}{f(x)}\frac{\mathrm{d}f}{\mathrm{d}x} > 0$, and hence, since $f(x) > 0$, we obtain $\frac{\mathrm{d}f}{\mathrm{d}x} < 0$. Therefore, $f(x)$ decreases as x increases. Now $\lim_{x\to\infty} f(x) = \left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\right) = e$. □

Lemma 2.3. *If $f : [1, \infty) \to \mathbb{R}$ is the function given by*

$$f(x) = \frac{e^{(x-1)/2}}{x}$$

for $x \geq 1$, then $f(x)$ increases to infinity as x increases from 2 to infinity.

Proof. Using differentiation, we obtain that the minimum value of f occurs at $x = 2$, and that f has no other turning points. Thus, $f(x)$ decreases from $f(1) = 1$ to $f(2) = e^{1/2}/2 < 1$ as x increases from 1 to 2, and $f(x)$ increases as x increases from 2. Since $f(x) = \frac{1}{x}\sum_{i=0}^{\infty}\frac{((x-1)/2)^i}{i!} = \frac{5}{8x} + \frac{1}{4} + \frac{x}{8} + \sum_{i=3}^{\infty}\frac{((x-1)/2)^i}{i!x}$, $f(x)$ increases from $f(2) < 1$ to infinity as x increases from 2 to infinity. □

Lemma 2.4. *Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon y \geq 2,\ 1 \leq x < 2y + 1\}$. Let $f : A \to \mathbb{R}$ be the function given by*

$$f(x,y) = \left(\frac{2y}{2y+1-x}\right)^{y} - x$$

for $(x, y) \in A$. For any $y_0 \in [2, \infty)$, $f(x_{y_0}, y_0) = 0$ for some unique $x_{y_0} \in (1, 2y_0 + 1)$, and $f(x, y) < 0$ for any $x \in (1, x_{y_0}]$ and $y \in [y_0, \infty)$ such that $x \neq x_{y_0}$ or $y \neq y_0$.

Moreover:

(a) If $y_0, y_1 \in [2, \infty)$ with $y_0 < y_1$, then $x_{y_0} < x_{y_1} < x_1$.

(b) For any real $\delta > 0$, there exists some $y_\delta \in [2, \infty)$ such that $x_y > x_1 - \delta$ for any $y \in (y_\delta, \infty)$.

Proof. Let $g : [1, 2y_0 + 1) \to \mathbb{R}$ such that $g(x) = f(x, y_0)$ for $x \in [1, 2y_0 + 1)$. We have

$$\frac{\mathrm{d}g}{\mathrm{d}x} = y_0\left(\frac{2y_0}{2y_0+1-x}\right)^{y_0-1}\frac{2y_0}{(2y_0+1-x)^2} - 1 = \frac{1}{2}\left(\frac{2y_0}{2y_0+1-x}\right)^{y_0+1} - 1.$$

As x increases from 1 to $2y_0 + 1$, the value of $\frac{1}{2}\left(\frac{2y_0}{2y_0+1-x}\right)^{y_0+1}$ increases from $\frac{1}{2}$ to ∞, and hence $\frac{\mathrm{d}g}{\mathrm{d}x}$ increases from $-\frac{1}{2}$ to ∞. Thus, there exists a unique $x^* \in (1, 2y_0 + 1)$ such that $\frac{\mathrm{d}g}{\mathrm{d}x}$ is 0 at x^*, and $g(x^*) = \min\{g(x)\colon x \in [1, 2y_0 + 1)\} < g(1) = 0$. Thus, $g(x)$ decreases from $g(1) = 0$ to $g(x^*)$, and then increases from $g(x^*)$ to ∞. Consequently, there exists a unique $x_{y_0} \in (1, 2y_0 + 1)$ such that $g(x_{y_0}) = 0 = g(1)$ and $g(x) < g(x_{y_0})$ for each $x \in (1, x_{y_0})$.

Now suppose $x \in (1, x_{y_0}]$ and $y \in [y_0, \infty)$. Let $z_0 = \frac{2y_0+1-x}{x-1}$ and $z = \frac{2y+1-x}{x-1}$. Then, $z \geq z_0$. We have

$$\begin{aligned} f(x,y) + x &= \left(1 + \frac{x-1}{2y+1-x}\right)^{y} = \left(1 + \frac{1}{z}\right)^{(z+1)(x-1)/2} \\ &= \left(\left(1 + \frac{1}{z}\right)^{z+1}\right)^{(x-1)/2} \leq \left(\left(1 + \frac{1}{z_0}\right)^{z_0+1}\right)^{(x-1)/2} \qquad \text{(by Lemma 2.2)} \\ &= f(x, y_0) + x. \end{aligned} \tag{8}$$

Therefore,

$$f(x,y) \leq f(x,y_0) = g(x) \leq g(x_{y_0}) = 0. \tag{9}$$

If $x \neq x_{y_0}$, then $x < x_{y_0}$, and hence $g(x) < g(x_{y_0})$. If $y \neq y_0$, then $y > y_0$, $z > z_0$, $\left(1 + \frac{1}{z}\right)^{z+1} < \left(1 + \frac{1}{z_0}\right)^{z_0+1}$ (by Lemma 2.2), and hence $f(x, y) < f(x, y_0)$ by (8). Thus, if $x \neq x_{y_0}$ or $y \neq y_0$, then $g(x) < g(x_{y_0})$ or $f(x, y) < f(x, y_0)$, and hence $f(x, y) < 0$ by (9).

Let $h : [1, \infty) \to \mathbb{R}$ such that $h(x) = e^{(x-1)/2} - x$ for $x \geq 1$. Using differentiation, we obtain that the minimum value of h occurs at $x = 1 + 2\ln 2 < x_1$, and that h has no other turning points. Thus, $h(x)$ decreases from $h(1) = 0$ to $h(1 + 2\ln 2) < 0$ as x increases from 1 to $1 + 2\ln 2$, and, since $h(x) = x\left(\frac{e^{(x-1)/2}}{x} - 1\right)$, Lemma 2.3 implies that $h(x)$ increases to infinity as x increases from $1 + 2\ln 2$ to infinity. Note that $h(x_1) = 0$. Let $z_0' = \frac{2y_0+1-x_{y_0}}{x_{y_0}-1}$. We have $0 = f(x_{y_0}, y_0) = \left(\left(1 + \frac{1}{z_0'}\right)^{z_0'+1}\right)^{(x_{y_0}-1)/2} - x_{y_0} > e^{(x_{y_0}-1)/2} - x_{y_0}$ by Lemma 2.2. Thus, $h(x_{y_0}) < 0$, and hence $x_{y_0} < x_1$.

Next, suppose $y_0 < y_1$. By the same argument for x_{y_0}, $x_{y_1} < x_1$. Let $p : [1, 2y_1 + 1) \to \mathbb{R}$ such that $p(x) = f(x, y_1)$ for $x \in [1, 2y_1 + 1)$. By the argument above, $p(x) < 0$ for $x \in (1, x_{y_1})$, $p(x_{y_1}) = 0$, and $p(x) > 0$ for $x \in (x_{y_1}, 2y_1 + 1)$. Since $y_1 \in (y_0, \infty)$, we have $p(x) = f(x, y_1) < 0$ for any $x \in (1, x_{y_0}]$, so $x_{y_1} > x_{y_0}$. Thus, (a) is proved.

Finally, let $\delta \in \mathbb{R}^+$. Let $\delta' = \min\{\frac{1}{2}, \delta\}$. Let $x' = x_1 - \delta'$. Since $x' < x_1$, $h(x') < 0$. Let $q : [2, \infty) \to \mathbb{R}$ such that $q(y) = f(x', y)$ for $y \geq 2$. We have $x' \geq x_1 - \frac{1}{2} > 3$ and $q(2) = \left(\frac{4}{5-x'}\right)^2 - x' > \left(\frac{4}{5-3}\right)^2 - x_1 > 0$. For any $y \in [2, \infty)$, let $z_y = \frac{2y+1-x'}{x'-1}$. As y increases to infinity, z_y increases to infinity. We have $q(y) = \left(\left(1 + \frac{1}{z_y}\right)^{z_y+1}\right)^{(x'-1)/2} - x'$. Thus, by Lemma 2.2, $q(y)$ decreases from $q(2) > 0$ to $h(x') < 0$ as y increases from 2 to infinity. Thus, there exists some $y_\delta \in [2, \infty)$ such that $q(y_\delta) = 0$. We have $f(x', y_\delta) = 0$, so $x' = x_{y_\delta}$. By (a), $x_y > x_{y_\delta}$ for any $y \in (y_\delta, \infty)$. Thus, (b) is proved. □

Lemma 2.5. *Let $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon y \geq 2,\ 1 \leq x < \frac{2y^2}{y-1} + 1\}$. Let $g : B \to \mathbb{R}$ be the function given by*

$$g(x, y) = \left(\frac{2y^2}{2y^2 - (y-1)(x-1)}\right)^y - x$$

for $(x, y) \in B$. For any $y_0 \in [2, \infty)$, $g(z_{y_0}, y_0) = 0$ for some unique $z_{y_0} \in (1, \frac{2{y_0}^2}{y_0-1} + 1)$, $g(x, y_0) < 0$ for any $x \in (1, z_{y_0})$, $g(x, y_0) > 0$ for any $x \in (z_{y_0}, \frac{2{y_0}^2}{y_0-1} + 1)$, and $z_{y_0} > x_{y_0}$, where x_{y_0} is as in Lemma 2.4.

Proof. By the same argument in the first paragraph of the proof of Lemma 2.4, $g(z_{y_0}, y_0) = 0$ for some unique $z_{y_0} \in (1, \frac{2{y_0}^2}{y_0-1} + 1)$, $g(x, y_0) < 0$ for any $x \in (1, z_{y_0})$, and $g(x, y_0) > 0$ for any $x \in (z_{y_0}, \frac{2{y_0}^2}{y_0-1} + 1)$. Let f be as in Lemma 2.4. By Lemma 2.4, $f(x_{y_0}, y_0) = 0$ for some unique $x_{y_0} \in (1, 2y_0 + 1)$, and $f(x, y) < 0$ for any $x \in (1, x_{y_0})$. We have $x_{y_0} < 2y_0 + 1 < \frac{2{y_0}^2}{y_0-1} + 1$. For any $x \in (1, x_{y_0}]$,

$$g(x, y_0) < \left(\frac{2{y_0}^2}{2{y_0}^2 - y_0(x-1)}\right)^{y_0} - x = f(x, y_0) \leq 0,$$

so $z_{y_0} > x_{y_0}$. □

Proof of Theorem 1.5 and of Proposition 2.1. Let $y_0 = k$. Let g and z_{y_0} be as in Lemma 2.5. Let $x_0 = z_{y_0}$. Let f and x_{y_0} be as in Lemma 2.4. By Lemma 2.5, $x_0 > x_{y_0}$. By Lemma 2.4 (a), the larger y_0 is, the larger x_{y_0} is. It can be checked that $x_{y_0} = 2.438...$ if $k = 2$. By Lemma 2.4 (b), for any real $\delta > 0$, $x_{y_0} > x_1 - \delta$ if k is sufficiently large. Proposition 2.1 follows.

Let $x = (k+1)t/n$. Since $n < (k+1)t$, $x > 1$. Obviously, $n \geq t \geq 1$, so $x \leq k + 1 < 2k + 1 < \frac{2k^2}{k-1} + 1$. Let $\sim$ be any of the relations $<$, $=$, and $>$. We have

$$b_1(k, t, n) \sim b_2(k, t, n) \quad \Leftrightarrow \quad \frac{n + (k-1)t}{n} \sim 2k\left(1 - \frac{k}{k+1}\left(\frac{n}{(k+1)t}\right)^{1/k}\right)$$

$$\Leftrightarrow \quad 1 + \frac{(k-1)x}{k+1} \sim 2k - \frac{2k^2}{(k+1)x^{1/k}} \quad \Leftrightarrow \quad \frac{2k^2}{x^{1/k}} \sim (2k-1)(k+1) - (k-1)x$$

$$\begin{aligned}
&\Leftrightarrow\quad (2k^2)^k \sim ((2k-1)(k+1)-(k-1)x)^k x > 0 \quad (\text{as } x \leq k+1)\\
&\Leftrightarrow\quad \left(\frac{2k^2}{2k^2-(k-1)(x-1)}\right)^k \sim x \quad \Leftrightarrow \quad g(x,y_0) \sim 0. \qquad (10)
\end{aligned}$$

Theorem 1.5 follows by Lemma 2.5. □

3 Proof of Theorem 1.6

This section is dedicated to the proof of Theorem 1.6.

By slightly modifying the function f in Lemma 2.4, we obtain the following lemma by the same argument for Lemma 2.4.

Lemma 3.1. *Let $A = \{(x,y) \in \mathbb{R} \times \mathbb{R} \colon y \geq 3,\, 1 \leq x < 2y\}$. Let $f : A \to \mathbb{R}$ be the function given by*

$$f(x,y) = \left(\frac{2y-1}{2y-x}\right)^{y-1/2} - x$$

for $(x,y) \in A$. For any $y_0 \in [3,\infty)$, $f(x_{y_0}, y_0) = 0$ for some unique $x_{y_0} \in (1, 2y_0)$, and $f(x,y) < 0$ for any $x \in (1, x_{y_0}]$ and $y \in [y_0, \infty)$ such that $x \neq x_{y_0}$ or $y \neq y_0$.

Moreover:
(a) If $y_0, y_1 \in [3,\infty)$ with $y_0 < y_1$, then $x_{y_0} < x_{y_1} < x_1$.
(b) For any real $\delta > 0$, there exists some $y_\delta \in [3,\infty)$ such that $x_y > x_1 - \delta$ for any $y \in (y_\delta, \infty)$.

Proof. Let $g : [1, 2y_0) \to \mathbb{R}$ such that $g(x) = f(x, y_0)$ for $x \in [1, 2y_0)$. We have

$$\frac{\mathrm{d}g}{\mathrm{d}x} = (y_0 - 1/2)\left(\frac{2y_0-1}{2y_0-x}\right)^{y_0-3/2}\frac{2y_0-1}{(2y_0-x)^2} - 1 = \frac{1}{2}\left(\frac{2y_0-1}{2y_0-x}\right)^{y_0+1/2} - 1.$$

As x increases from 1 to $2y_0$, the value of $\frac{1}{2}\left(\frac{2y_0-1}{2y_0-x}\right)^{y_0+1/2}$ increases from $\frac{1}{2}$ to ∞, and hence $\frac{\mathrm{d}g}{\mathrm{d}x}$ increases from $-\frac{1}{2}$ to ∞. Thus, there exists a unique $x^* \in (1, 2y_0)$ such that $\frac{\mathrm{d}g}{\mathrm{d}x}$ is 0 at x^*, and $g(x^*) = \min\{g(x) \colon x \in [1, 2y_0)\} < g(1) = 0$. Thus, $g(x)$ decreases from $g(1) = 0$ to $g(x^*)$, and then increases from $g(x^*)$ to ∞. Consequently, there exists a unique $x_{y_0} \in (1, 2y_0)$ such that $g(x_{y_0}) = 0 = g(1)$ and $g(x) < g(x_{y_0})$ for each $x \in (1, x_{y_0})$.

Now suppose $x \in (1, x_{y_0}]$ and $y \in [y_0, \infty)$. Let $z_0 = \frac{2y_0-x}{x-1}$ and $z = \frac{2y-x}{x-1}$. Then, $z \geq z_0$. We have

$$\begin{aligned}
f(x,y) + x &= \left(1 + \frac{x-1}{2y-x}\right)^{y-1/2} = \left(1 + \frac{1}{z}\right)^{(z+1)(x-1)/2}\\
&= \left(\left(1+\frac{1}{z}\right)^{z+1}\right)^{(x-1)/2} \leq \left(\left(1+\frac{1}{z_0}\right)^{z_0+1}\right)^{(x-1)/2} \qquad (\text{by Lemma 2.2})\\
&= f(x,y_0) + x. \qquad (11)
\end{aligned}$$

By following the proof of Lemma 2.4 from (8) onwards, we obtain Lemma 3.1. □

Proof of Theorem 1.6. By (4), $m \leq kt$, and equality holds if G is the union of t pairwise vertex-disjoint $(k+1)$-vertex stars, in which case the bounds in Theorems 1.3 and 1.4 are equal and attained. We now consider $m < kt$.

If $k = 2$, then

$$b_4(k,t,m) - b_3(k,t,m) = m - \frac{m^2}{4t} - \frac{m+t}{3} = \frac{(2t-m)(3m-2t)}{12t} > 0$$

as $m < kt = 2t$ and $m \geq kt/2 = t$ by (5). Thus, $b_3(k,t,m) < b_4(k,t,m)$ if $k = 2$.

We now consider $k \geq 3$. Let $x = kt/m$. Since $m < kt$, $x > 1$. By (5), $m \geq kt/2$, so $x \leq 2$. Let $\sim$ be any of the relations $<$, $=$, and $>$. We have

$$\begin{aligned}
& b_3(k,t,m) \sim b_4(k,t,m) \quad \Leftrightarrow \quad \frac{m+(k-1)t}{m} \sim (2k-1)\left(1 - \frac{k-1}{k}\left(\frac{m}{kt}\right)^{1/(k-1)}\right) \\
\Leftrightarrow \quad & 1 + (k-1)\frac{x}{k} \sim 2k - 1 - \frac{(2k-1)(k-1)}{kx^{1/(k-1)}} \\
\Leftrightarrow \quad & \frac{(2k-1)(k-1)}{x^{1/(k-1)}} \sim 2k(k-1) - (k-1)x \quad \Leftrightarrow \quad \frac{(2k-1)}{x^{1/(k-1)}} \sim 2k - x \\
\Leftrightarrow \quad & (2k-1)^{k-1} \sim (2k-x)^{k-1}x > 0 \quad (\text{as } 1 < x \leq 2) \\
\Leftrightarrow \quad & \left(\frac{2k-1}{2k-x}\right)^{k-1} \sim x. \qquad (12)
\end{aligned}$$

Let f be as in Lemma 3.1. Let $y_0 = k$. By Lemma 3.1, $f(x_{y_0}, y_0) = 0$ for some unique $x_{y_0} \in (1, x_1)$, and the larger y_0 is, the larger x_{y_0} is. Let $x_0 = x_{y_0}$. It can be checked that $x_0 = 2.575...$ if $k = 3$. Since $m \geq kt/2$, $m > kt/x_0$. Thus, $x < x_0$. We have $\left(\frac{2k-1}{2k-x}\right)^{k-1/2} - x = f(x, y_0) < 0$ by Lemma 3.1. Since $\left(\frac{2k-1}{2k-x}\right)^{k-1} < \left(\frac{2k-1}{2k-x}\right)^{k-1/2} < x$, we have $b_3(k,t,m) < b_4(k,t,m)$ by (12). □

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